Trigonometric functions of half- angle

Formulas are:

1. $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ or $2\sin^2\frac{\alpha}{2} = 1-\cos\alpha$
2. $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ or $2\cos^2\frac{\alpha}{2} = 1+\cos\alpha$
3. $tg\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$
4. $ctg\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$

Examples:

1) Find $\cos\frac{\alpha}{2}$**, if** $\sin\alpha = \frac{4}{5}$ **and** $\alpha \in \left(-\frac{3\pi}{2}, -\pi\right)$**.**

Solution:

Because: $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$, we must find $\cos\alpha$.

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\cos^2\alpha = 1-\sin^2\alpha$$
$$\cos^2\alpha = 1-\left(\frac{4}{5}\right)^2$$
$$\cos^2\alpha = 1-\frac{16}{25}$$
$$\cos^2\alpha = \frac{9}{25}$$
$$\cos^2\alpha = \pm\sqrt{\frac{9}{25}}$$
$$\cos\alpha = \pm\frac{3}{5}$$

We have that $\alpha \in \left(-\frac{3\pi}{2}, \pi\right)$

This tells us that angle is from the II quarter.

$$\cos\alpha = -\frac{3}{5}$$

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1-\frac{3}{5}}{2}}$$
$$\cos\frac{\alpha}{2} = -\sqrt{\frac{\frac{2}{5}}{2}}$$
$$\cos\frac{\alpha}{2} = -\frac{1}{\sqrt{5}}$$
$$\cos\frac{\alpha}{2} = -\frac{\sqrt{5}}{5}$$

$$\alpha \in \left(-\frac{3\pi}{2}, -\pi\right) \Rightarrow \frac{\alpha}{2} \in \pi\left(-\frac{3\pi}{4}, \frac{\pi}{2}\right)$$

2) Find $\cos\frac{\alpha}{2}$, if $\sin\alpha = \frac{4\sqrt{2}}{9}$ and $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$.

Solution:

First, we must find $\cos\alpha$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

$$\cos^2\alpha = 1 - \left(-\frac{4\sqrt{2}}{9}\right)^2$$

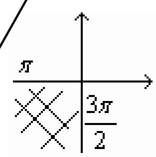
$$\cos^2\alpha = 1 - \frac{16\cdot 2}{81}$$

$$\cos^2\alpha = 1 - \frac{32}{81}$$

$$\cos^2\alpha = \frac{49}{81}$$

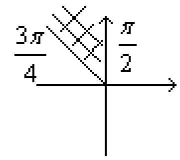
$$\cos\alpha = \pm\frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow$$



$$\cos\alpha = -\frac{7}{9}$$

$$\alpha \in \left(\pi, \frac{3\pi}{2}\right) \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$



For $\sin\frac{\alpha}{2}$ we must take +

$$\sin\frac{\alpha}{2} = +\sqrt{\frac{1-\cos\alpha}{2}} = +\sqrt{\frac{1+\frac{7}{9}}{2}}$$

$$\sin\frac{\alpha}{2} = +\frac{4}{3\sqrt{2}}$$

$$\sin\frac{\alpha}{2} = \frac{4}{3\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{3\cdot 2}$$

$$\sin\frac{\alpha}{2} = \frac{2\sqrt{2}}{3}$$

3) Prove that:

a) $tg\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}$ for $\alpha \neq \pi(2k+1), k \in z$

b) $tg\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$ for $\alpha \neq \pi(2k+1), k \in z$

Solution:

a) In proof we will use: $\left[\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]$

$$\frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = tg\frac{\alpha}{2}$$

b) $$\frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = tg\frac{\alpha}{2}$$

4) Calculated the value of expression $A=\dfrac{\sin x+2\cos x}{tgx-ctgx}$ **, if** $tg\dfrac{x}{2}=2$

Solution:

First , we will use $tg\dfrac{x}{2}=2$, and find $\cos x$

$$tg\frac{x}{2}=\sqrt{\frac{1-\cos x}{1+\cos\alpha}}$$

$$\sqrt{\frac{1-\cos x}{1+\cos\alpha}}=2$$

$$\frac{1-\cos x}{1+\cos\alpha}=4$$

$$1-\cos x=4(1+\cos)$$

$$1-\cos x=4+4\cos x$$

$$-\cos x-4\cos x=4-1$$

$$-5\cos x=3$$

$$\cos x=-\frac{3}{5}$$

$$\sin^2 x=1-\cos^2 x$$

$$\sin^2 x=1-\frac{9}{25}$$

$$\sin^2 x=\frac{16}{25}$$

$$\sin x=\pm\frac{4}{5}$$

Angle is from the II quarter.

$$\sin x=+\frac{4}{5}$$

$$tgx=\frac{\sin x}{\cos x}=\frac{\frac{4}{5}}{-\frac{3}{5}}=-\frac{4}{3}$$

$$ctgx=-\frac{3}{4}$$

A=?

$$A=\frac{\frac{4}{5}+2\cdot\left(-\frac{3}{5}\right)}{-\frac{4}{3}+\frac{3}{4}}$$

$$A=\frac{\frac{-2}{5}}{\frac{-7}{12}}$$

$$A=\frac{24}{35}$$

5) Prove:

a) $\frac{1+\sin x-\cos x}{1+\sin x+\cos x}=tg\frac{x}{2}$

b) $\frac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=tg^2\frac{x}{2}$

Solution:

a)

$$\frac{1+\sin x-\cos x}{1+\sin x+\cos x}= \quad \text{The idea is :} \quad \begin{cases} 1-\cos x=2\sin^2\frac{x}{2} \\ 1+\cos x=2\cos^2\frac{x}{2} \\ \sin x=2\sin\frac{x}{2}\cos\frac{x}{2} \end{cases}$$

$$=\frac{2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}=$$

$$=\frac{2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}{2\cos\frac{x}{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=tg\frac{x}{2}$$

b)

$$\frac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=\frac{2\sin x-2\sin x\cos x}{2\sin x+2\sin x\cos x}=$$

$$=\frac{2\sin x(1-\cos x)}{2\sin x(1+\cos x)}=$$

$$=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}=tg^2\frac{x}{2}$$

6) Figure without the use of mathematical tools : $tg7^o30^{'}$ =?

Solution:

The idea is to use just proven equality: $tg\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$

$$\text{So:} \qquad tg7^o30^{'} = tg\frac{15^o}{2} = \frac{\sin 15^o}{1+\cos 15^o}$$

Now , we have to find $\sin 15^o$ and $\cos 15^o$

$$\sin 15^o = \sin\frac{30^o}{2} = \sqrt{\frac{1-\cos 30^o}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} \longrightarrow \sin 15^o = \frac{\sqrt{2-\sqrt{3}}}{2}$$

This is true, but it's a little complicated……

This can be done in more convenient way:

$$\sin 15^o = \sin(45^o - 30^o) = \sin 45^o \cos 30^o - \cos 45^o \sin 30^o =$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

$$\cos 15^o = \cos(45^o - 30^o) = \cos 45^o \cos 30^o - \sin 45^o \sin 30^o =$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$$

$$tg7^o30^{'} = \frac{\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}}{1+\frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}}$$

$$tg7^o30^{'} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}+5} = \text{ double rationalization (see in the root)}$$

$$tg7^o30^{'} = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$$

7) If $tg\frac{x}{2}=t$,calculated $\sin x$, $\cos x$ and tgx using t.

Solution:

This will be a replasement in trigonometric integrals, so pay attention!

$$\sin x=\frac{\sin x}{1}=\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cancel{\cos^2\frac{x}{2}}\left(2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)}{\cancel{\cos^2\frac{x}{2}}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1\right)}$$

$$=\frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1}=\frac{2t}{t^2+1}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{\cos x}{1}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{\cancel{\cos^2\frac{x}{2}}\left(1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)}{\cancel{\cos^2\frac{x}{2}}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}+1\right)}$$

$$=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-t^2}{1+t^2}$$

$$tgx=\frac{\sin x}{\cos x}=\frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}=\frac{2t}{1-t^2}$$

8) Calculate $\dfrac{\sin 160^o}{\cos^4 40^o - \sin^4 40^o} = ?$

Solution:

$$\frac{\sin 160^o}{\cos^4 40^o - \sin^4 40^o} = \frac{\sin 160^o}{\underbrace{(\cos^2 40^o + \sin^2 40^o)}_{1}(\cos^2 40^o - \sin^2 40^o)} =$$

$$\frac{\sin 160^o}{\cos^2 40^o - \sin^2 40^o} = \{\text{this is down formula } \cos^2 - \sin^2 x = \cos 2x\}$$

$$\frac{\sin 160^o}{\cos 80^o} = \frac{2\sin 80^o \cancel{\cos 80^o}}{\cancel{\cos 80^o}} = 2\sin 80^o$$